SEIBERG-WITTEN INVARIANTS OF 4-MANIFOLDS WITH FREE CIRCLE ACTIONS

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1. Introduction

The main result of this paper describes a formula for the Seiberg-Witten invariant of a 4-manifold X which admits a nontrivial free S^1-action. A free circle action on X is classified by its orbit space, a 3-manifold M, and its Euler class $\chi \in H^2(M;\mathbb{Z})$. If $\chi = 0$, then $X = M \times S^1$, and it is well-known that the Seiberg-Witten invariants of X are equal to the 3-dimensional Seiberg-Witten invariants of M.

Our result expresses the Seiberg-Witten invariants of X are in terms of the Seiberg-Witten invariants of M and the Euler class χ:

Theorem 1. *Let X be a smooth 4-manifold with $b_+ \geq 2$ and a free circle action. Let M^3 be the smooth orbit space and suppose that the Euler class $\chi \in H^2(M;\mathbb{Z})$ of the free circle action is not torsion. Let ξ be a* spin^c *structure over X. If ξ is not pulled up via $\pi : X \to M$, then $SW_X(\xi) = 0$. Otherwise, let ξ^* be a* spin^c *structure on M such that $\xi = \pi^*(\xi^*)$, then*

$$SW_X^4(\xi) \;=\; \sum_{\xi' \equiv \xi^* \mod \chi} SW_M^3(\xi'). \qquad (1)$$

The difference of two spin^c structures gives rise to a well-defined element $\xi' - \xi \in H^2(X;\mathbb{Z})$. For more information, see section (4.1). Because χ is nontorsion, the equivalence relation in the above theorem is well-defined. The pullback of a spin^c structure is discussed in section (4.2).

As an application of this theorem we shall produce a nonsymplectic 4-manifold with a free circle action whose orbit space fibers over S^1. This example runs counter to intuition since there is a well-known conjecture of Taubes that $M^3 \times S^1$ admits a symplectic structure if and only if M^3 fibers over the S^1. Furthermore, there is evidence [FGM] which suggests that many such 4-manifolds are, in fact, symplectic. As another application of our formula, we construct a 3-manifold which

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is not the orbit space of any symplectic 4-manifold with a free circle action. A corollary of the main theorem is a formula for the Seiberg-Witten invariant of the total space of a circle bundle over a surface. This formula can be thought of as the 3 dimensional analog of the 4 dimensional formula.

2. Classifying free circle actions

Let X be an oriented connected 4-manifold carrying a smooth free S^1-action. Its orbit space M is a 3-manifold whose orientation is determined, so that, followed by the natural orientation on the orbits, the orientation of X is obtained. Choose a smooth connected loop l representing the the Poincaré dual $PD(\chi) \in H_1(M;\mathbb{Z})$. Remove a tubular neighborhood $N \cong D^2 \times l$ of l from M, and set $X_0 = (M \setminus N) \times S^1$. View X_0 as an S^1-manifold whose action is given by rotation in the last factor. Let m be the meridian of l, and let t be an orbit in X_0. We then have:

Lemma 2. *The manifold X is diffeomorphic (by a bundle isomorphism) to the manifold*

$$X(l) = X_0 \cup_\varphi D^2 \times T^2 \tag{2}$$

where $\varphi : T^3 \to \partial X_0$ is an equivariant diffeomorphism which evaluates $\varphi_*([\partial(D^2 \times pt)] = [m+t]$ in homology.

When gluing $D^2 \times T^2$ into the boundary of a manifold, the resulting closed manifold is determined up to diffeomorphism by the image in homology of $[\partial(D^2 \times pt)]$. (For example, see [MMS].)

Proof. The manifold X is a principal S^1-bundle. Since χ evaluates on any 2-cycle in $M \setminus N$ by intersecting that 2-cycle against l, it follows that the restriction of the Euler class χ restricts trivially to $M \setminus N$. Therefore, the S^1-bundle is trivial over $M \setminus N$, and $\pi^{-1}(M \setminus N)$ is diffeomorphic to X_0. Similarly, $\pi^{-1}(N)$ is diffeomorphic to $D^2 \times S^1 \times S^1$. Let m', l', and t' be the circles which correspond to the factors in $D^2 \times S^1 \times S^1$ respectively.

Construct a manifold $X(l)$ as above using a bundle isomorphism $\varphi : \partial(D^2 \times S^1) \times S^1 \to X_0$. Bundle isomorphisms covering the identity are classified up to vertical equivariant isotopy by homotopy classes of maps in $[\partial(D^2 \times S^1), S^1] = \mathbb{Z} \oplus \mathbb{Z}$. Explicitly, an equivariant map φ inducing $1_{\partial(D^2 \times S^1)}$ is classified by integers (r,s) where $\varphi_*[m'] = [m] + r[t]$ and $\varphi_*[l'] = [l] + s[t]$. A bundle automorphism Φ of $(D^2 \times S^1) \times S^1$ can be constructed such that $\Phi_*[m'] = [m']$ and $\Phi_*[l'] = [l'] + s[t']$ for any $s \in \mathbb{Z}$. These bundle automorphisms are just the equivariant maps

classified by $[D^2 \times S^1, S^1] = H^1(D^2 \times S^1; \mathbb{Z})$. Therefore the resulting bundle $X(l)$ depends only on the integer r and the homology class $[l]$. In particular, the obstruction to extending the constant section

$$M \setminus N \to X_0 = (M \setminus N) \times S^1$$

over $D^2 \times S^1$ lies in $H^2(D^2 \times S^1, \partial(D^2 \times S^1); \mathbb{Z})$ and is given by r. The Euler class of $X(l)$ is then $PD(r[l]) = r\chi$. Taking $r = 1$ produces the desired bundle. □

From now on we shall work with $X(l)$ and refer to it as X. Furthermore, it is clear from the construction above that the map φ can be chosen so that in homology,

$$\varphi_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{3}$$

with respect to the basis $\{[m], [l], [t]\}$.

3. Gluing along T^3

Since we have $X = X_0 \cup_\varphi (D^2 \times T^2)$ we may apply the gluing theorem of Morgan, Mrowka, and Szabó [MMS]. Recall that $\varphi_*([m']) = [m+t]$.

Theorem 3 (Morgan, Mrowka, and Szabó)**.** *If the* spin^c *structure ξ over X restricts nontrivially to $D^2 \times T^2$, then $SW_X(\xi) = 0$. For each* spin^c *structure $\xi_0 \to X_0$ that restricts trivially to ∂X_0, let $V_X(\xi_0)$ denote the set of isomorphism classes of* spin^c *structures over X whose restriction to X_0 is equal to ξ_0. Then we have*

$$\sum_{\xi \in V_X(\xi_0)} SW_X(\xi) = \sum_{\xi \in V_{M\times S^1}(\xi_0)} SW_{M\times S^1}(\xi) + \sum_{\xi \in V_{X_{0/1}}(\xi_0)} SW_{X_{0/1}}(\xi), \tag{4}$$

where the manifold $X_{0/1} = X_0 \cup_{\varphi_{0,1}} D^2 \times T^2$ is defined by the map $\varphi_{0,1}$ which maps $[m'] \mapsto [t]$ in homology.

In our situation, this formula simplifies significantly. Let i denote the inclusion of ∂X_0 into X_0. A study of the long exact sequences in homology shows that the left hand side consists of a single term when $i_*[m+t]$ is indivisible. Since $i_*[t]$ is independent of $i_*[m]$ and $i_*[t]$ is a primitive class in $H_1(X_0; \mathbb{Z})$, $i_*[m+t]$ is such a class. Therefore, the formula enables the calculation of the SW invariants of X in terms of the SW invariants of $M \times S^1$ and a manifold $X_{0/1}$.

The manifold $X_{0/1}$ admits a semi-free S^1-action whose fixed point set is a torus. Its orbit space is $M \setminus N$, and $\partial(M \setminus N) = \partial N$ is the

image of the fixed point set. The condition $b_+(X) \geq 2$ of the main theorem implies that $b_+(X_{0/1}) > 1$ and that

$$\text{rank } H_1(M \setminus N, \partial(M \setminus N); \mathbb{Z}) > 1.$$

The two statements are proved as follows. The Gysin sequence

$$H^2(M;\mathbb{Z}) \xrightarrow{\pi^*} H^2(X;\mathbb{Z}) \longrightarrow H^1(M;\mathbb{Z}) \xrightarrow{\cdot\cup\chi} H^3(M;\mathbb{Z}) \tag{5}$$

implies

$$H^2(X;\mathbb{Z}) \cong \big(H^2(M;\mathbb{Z})/<\chi>\big) \\ \oplus \ker\big(\cup\chi : H^1(M;\mathbb{Z}) \to H^3(M;\mathbb{Z})\big). \tag{6}$$

Each component of the direct sum above has rank $b_1(M) - 1$. The bilinear form of X is the direct sum of hyperbolic pairs which implies that $b_+(X) = b_1(M) - 1$. Since $[l]$ is not a torsion element, removing N from M implies the rank of $H_1(M \setminus N, \partial(M \setminus N); \mathbb{Z})$ is also $b_1(M) - 1$. The second statement now follows because $b_1(M) - 1 = b_+(X) > 1$. The first statement requires the following Mayer-Vietoris sequence

$$H_3(T^3;\mathbb{Z}) \to H_2(X_0;\mathbb{Z}) \oplus H_2(D^2 \times T^2;\mathbb{Z}) \to H_2(X_{0/1};\mathbb{Z}) \xrightarrow{0} H_1(T^3;\mathbb{Z}).$$

The rank of $H_2(X_0;\mathbb{Z})$ is $2b_1(M) - 1$ and the rank of the image of the first map is 2. Therefore $b_2(X_{0/1}) = 2b_1(M) - 2$. Since the bilinear form of $X_{0/1}$ is also a direct sum of hyperbolic pairs, $b_+(X_{0/1}) > 1$.

Proposition 4. *Let X be a smooth closed oriented 4-manifold with a smooth semi-free circle action and $b_+(X) > 1$. Let $X^* = X/S^1$ be its orbit space. Suppose that X^* has a nonempty boundary and* rank $H_1(X^*, \partial X^*; \mathbb{Z}) > 1$. *Then $SW_X \equiv 0$.*

Proof. Let F denote the fixed point set of X and F^* its image in X^*. Then $\partial X^* \subset F^*$. The restriction of the circle action to $X \setminus F$ defines a principal S^1-bundle whose Euler class lies in $H^2(X^* \setminus F^*; \mathbb{Z})$. Let $\chi' \in H_1(X^*, F^*; \mathbb{Z})$ denote its Poincaré dual. Consider the exact sequence

$$0 \to H_1(X^*, \partial X^*; \mathbb{Z}) \xrightarrow{i_*} H_1(X^*, F^*; \mathbb{Z}) \to \\ \to H_0(F^*, \partial X^*; \mathbb{Z}) \to H_0(X^*, \partial X^*; \mathbb{Z}).$$

Since the rank of $H_1(X^*, \partial X^*; \mathbb{Z})$ is greater than 1, there is a class in $i_*(H_1(X^*, \partial X^*; \mathbb{Z}))$ which is primitive and not a multiple of χ'. This class may be represented by a path α in X^* which starts and ends on ∂X but is otherwise disjoint from F^*.

The preimage $S = \pi^{-1}(\alpha)$ is a 2-sphere of self-intersection 0 in X. The Gysin sequence gives:

$$H_3(X^*, F^*, \mathbb{Z}) \to H_1(X^*, F^*, \mathbb{Z}) \xrightarrow{\rho} H_2(X, F, \mathbb{Z}) \to H_2(X^*, F^*, \mathbb{Z})$$

where $\rho_*(i_*[\alpha]) = [S]$. The image of $H_3(X^*, F^*, \mathbb{Z}) \cong \mathbb{Z}$ in $H_1(X^*, F^*, \mathbb{Z})$ is generated by χ'. Since $i_*[\alpha]$ is primitive and not a multiple of χ', the class $[S] \in \mathrm{Im}\rho \subset H_2(X, F, \mathbb{Z})$ is not torsion; hence $[S]$ is nontorsion as an element of $H_2(X; \mathbb{Z})$.

It now follows from [FS1] that $\mathrm{SW}_X \equiv 0$. $\square$

This type of vanishing theorem is quite common for 4-manifolds with circle actions. For instance, it follows from [F] that Seiberg-Witten invariants vanish for simply connected 4-manifolds which have $b_+ > 1$ and a smooth circle action.

Proposition 4 implies that the formula (4) simplifies to

$$SW_X(\xi) \;=\; \sum_{\xi' \in V_{M\times S^1}(\xi|_{X_0})} SW_{M\times S^1}(\xi'). \tag{7}$$

4. Understanding the spin^c structures

In this section we shall prove that all basic classes of X come from spin^c structures that are pulled up from M (in a suitable sense). We shall also identify the spin^c structures in the set $V_{M\times S^1}(\xi|_{X_0})$ coming from the gluing theorem.

4.1. **Spin^c structures.** First recall some basic facts about spin^c structures. The set of spin^c structures lifting the frame bundle of a 4-manifold X is a principal homogeneous space over $H^2(X; \mathbb{Z})$: given two spin^c structures ξ_1, ξ_2 their difference $\delta(\xi_1, \xi_2)$ is a well-defined element of $H^2(X; \mathbb{Z})$. For details, see [FM] or [R].

Likewise, if ξ is a spin^c structure and $e \in H^2(X; \mathbb{Z})$ is a 2-dimensional cohomology class, there is a new spin^c structure $\xi + e$. Let W_ξ be spinor bundle associated with ξ, then the new spinor bundle is $W_\xi \otimes L_e$ where L_e is the unique line bundle with first Chern class e.

For all spin^c structures, a line bundle L_ξ can be associated to ξ called the determinant line bundle. Let (ξ, L_ξ) be a pair consisting a spin^c structure ξ whose determinant line bundle is L_ξ. Given two spin^c structures (ξ_1, L_1), (ξ_2, L_2), the difference of their determinant line bundles is $c_1(L_1) - c_1(L_2) = 2e$ for some element $e \in H^2(X; \mathbb{Z})$. If $H^2(X; \mathbb{Z})$ has no 2-torsion, then e is well-defined and $c_1(L_\xi)$ determines the spin^c structure for (ξ, L_ξ). When $H^2(X; \mathbb{Z})$ has 2-torsion, one has a choice of two or more possible square roots of $2e$ and it seems that e is not well-defined. However, the difference element $\delta(\xi_1, \xi_2)$ satisfies

$c_1(L_1)-c_1(L_2) = 2\delta(\xi_1, \xi_2)$ and so there is a unique element in $H^2(X;\mathbb{Z})$ which determines the difference of two spin^c structures even in the presence of 2-torsion. So while $c_1(L_\xi)$ does not determine ξ in this case, the difference between two spin^c structures is still well-defined.

4.2. **Pullbacks of spin^c structures.** The spin^c structures on a 3-manifold M are defined by a pair $\xi = (W, \rho)$ consisting of a rank 2 complex bundle W with a hermitian metric (the spinor bundle) and an action ρ of 1-forms on spinors,

$$\rho : T^*M \to \mathrm{End}(W),$$

which satisfies the following property

$$\rho(v)\rho(w) + \rho(w)\rho(v) = -2 < v, w > \mathbf{Id}_W.$$

For a 4-manifold the definition is similar, but consists of a rank 4 complex bundle with an action on the cotangent space that satisfies the same property. There is a natural way to define the pullback of a spin^c structure. Let η denote the connection 1-form of the circle bundle $\pi : X \to M$, and let g_M be a metric on M, then we can endow X with the metric $g_X = \eta \otimes \eta + \pi^*(g_M)$. Using this metric, there is an orthogonal splitting

$$T^*X \cong \mathbb{R}\eta \oplus \pi^*(T^*M).$$

If $\xi = (W, \rho)$ is a spin^c structure over M, define the pullback of ξ to be $\pi^*(\xi) = (\pi^*(W) \oplus \pi^*(W), \sigma)$ where the action

$$\sigma : T^*X \to \mathrm{End}(\pi^*(W) \oplus \pi^*(W))$$

is given by

$$\sigma(b\eta + \pi^*(a)) = \begin{pmatrix} 0 & \pi^*(\rho(a)) + b\mathbf{Id}_{\pi^*}(W) \\ \pi^*(\rho(a)) - b\mathbf{Id}_{\pi^*(W)} & 0 \end{pmatrix}.$$

One can easily check that this defines a spin^c structure on X. Note that the first Chern class of $\pi^*(\xi)$ is just $\pi^*(c_1(L_\xi))$. The other pulled back spin^c structures are now obtained by the addition of classes $\pi^*(e)$ for $e \in H^2(M;\mathbb{Z})$.

There are spin^c structures on X which do not arise from spin^c structures that are pulled up from M. In the next section we show that the Seiberg-Witten invariants vanish for these spin^c structures.

4.3. **Spin^c structures which are not pullbacks.** Fix a spin^c structure $\xi_0 = (W_0, \rho)$ on M and consider its pullback $\xi = \pi^*(\xi_0)$ over X. Looking at the Gysin sequence (5), if a class $e \in H^2(X;\mathbb{Z})$ is not in the image of π^*, then $\xi + e$ is not a spin^c structure which is pulled back from M.

Lemma 5. *If (ξ, L_ξ) is a* spin^c *structure on X which is not pulled back from M, then $SW_X(\xi) = 0$.*

Proof. We claim that there exists an embedded torus which pairs nontrivially with $c_1(L_\xi)$. Then by the adjunction inequality [KM] the spin^c structure ξ has Seiberg-Witten invariant equal to zero. Let

$$\mathbf{H} = \ker(\cdot \cup \chi : H^1(M;\mathbb{Z}) \to H^3(M;\mathbb{Z}))$$

in equation (6), and consider for a moment the projection of $c_1(L_\xi)$ onto the first factor of $\mathbf{H} \oplus \pi^*(H^2(M;\mathbb{Z}))$ by changing the spin^c structure by an element of $\pi^*(H^2(M;\mathbb{Z}))$. Since ξ is not pulled back from M, $c_1(L_\xi)|_{\mathbf{H}} \neq 0$, and since $H^1(M;\mathbb{Z})$ is a free abelian group, $c_1(L_\xi)|_{\mathbf{H}}$ is not a torsion class.

Examining the Gysin sequence, $c_1(L_\xi)|_{\mathbf{H}} \in H^2(X;\mathbb{Z})$ maps to a class $\beta \in H^1(M;\mathbb{Z})$, $\beta \cup \chi = 0$. Thus the Poincaré dual of β can be represented by a surface b, and there is a 1-cycle λ in $M \setminus N$ rel ∂ such that $[\lambda] \cdot [b] \neq 0$. Since ∂N is connected, $[\lambda]$ is actually represented by a loop λ in $M \setminus N$. The preimage $\pi^{-1}(\lambda) = \lambda \times S^1$ in X is a torus, and $c_1(L_\xi)|_{\mathbf{H}} \cdot [\pi^{-1}(\lambda)] = [b] \cdot [\lambda] \neq 0$.

On the other hand, if $A \in \pi^* H^2(M;\mathbb{Z})$ then it Poincaré dual is represented by a loop α in M which may be chosen disjoint from λ. Thus $A \cdot [\pi^{-1}(\lambda)] = 0$. This means that $c_1(L_\xi) \cdot [\pi^{-1}(\lambda)] \neq 0$, as required. □

4.4. **Identifying the set $V_{M\times S^1}(\xi|_{X_0})$.** According to the previous lemma, the only nontrivial Seiberg-Witten spin^c structures are those pulled up from M. Thus far we have seen that for such a spin^c structure $\xi = \pi^*(\xi^*)$ with $\xi_0 = \xi|_{X_0}$, we have

$$SW_X(\xi) = \sum_{\xi' \in V_{M\times S^1}(\xi_0)} SW_{M\times S^1}(\xi').$$

Let $\tilde{\pi} : M \times S^1 \to M$ be the projection. We identify the set $V_{M\times S^1}(\xi_0)$ of isomorphism classes of spin^c structures over $M \times S^1$ which restrict on X_0 to ξ_0.

Lemma 6. $V_{M\times S^1}(\xi_0) = \{\, \tilde{\pi}^*(\xi^* + n\cdot\chi) \mid n \in \mathbb{Z} \,\}$.

Proof. The diagram

$$\begin{tikzcd}
X \arrow[drr, "\pi"'] & & X_0 \arrow[ll, "inc"'] \arrow[rr, "inc"] \arrow[d, "\tilde{\pi}|_{M \setminus N}"] & & M \times S^1 \arrow[dll, "\tilde{\pi}"] \\
& & M \setminus N \arrow[d, "inc"] & & \\
& & M & &
\end{tikzcd}$$

induces spin^c structures on X, X_0, and $M \times S^1$ which satisfy

$$inc^*(\pi^*(\xi^*)) = \xi_0 = inc^*(\tilde{\pi}^*(\xi^*).$$

Recall that ξ is the only spin^c structure induced on X by ξ_0 since $i_*[m + t]$ is indivisible. Since $\tilde{\pi}^*(\xi^*) \in V_{M\times S^1}(\xi_0)$, the set of spin^c structures on $M\times S^1$ is $\{\tilde{\pi}^*(\xi^*)+e | e \in H^2(M\times S^1;\mathbb{Z})\}$. Now $\tilde{\pi}^*(\xi^*)+e$ lies in $V_{M\times S^1}(\xi_0)$ if and only if $inc^*(\pi^*(\xi^*) + e) = \xi_0$, i.e. if and only if $inc^*(e) = 0$. Therefore,

$$V_{M\times S^1}(\xi_0) = \{\tilde{\pi}^*(\xi^*) + e \mid inc^*(e) = 0\}\,. \tag{8}$$

The kernel of inc^* is equal to the image of j^* in the diagram below.

$$\begin{tikzcd}
H^2(M \times S^1, (M \setminus N) \times S^1; \mathbb{Z}) \arrow[r, "j^*"] \arrow[d, "PD"] & H^2(M \times S^1; \mathbb{Z}) \arrow[r, "inc^*"] \arrow[d, "PD"] & H^2(X_0; \mathbb{Z}) \arrow[d, "PD"] \\
H_2(D^2 \times T^2; \mathbb{Z}) \arrow[r] & H_2(M \times S^1; \mathbb{Z}) \arrow[r] & H_2(X_0, \partial X_0; \mathbb{Z})
\end{tikzcd}$$

$$\begin{tikzcd}
n[T^2] \arrow[r, "j_*"'] & n[l \times t] \arrow[r] & 0
\end{tikzcd}$$

However $j_*[\mathrm{pt}\times T^2] = [l\times t]$, and since $\tilde{\pi}^*(\chi) = PD^{-1}[l\times t]$, the lemma follows. □

4.5. **Relationship between SW^3 and SW^4.** The following is a well-known fact about the relationship between the 3-dimensional Seiberg-Witten invariants and the 4-dimensional invariants.

Proposition 7 (cf. Donaldson [D])**.** *After making a suitable choice of orientations for M and $M \times S^1$, the following equality holds*

$$SW^3_M(\xi) = SW^4_{M\times S^1}(\tilde{\pi}^*(\xi))$$

for a spin^c *structure ξ over M.*

A natural choice of orientations for $M \times S^1$ and M is induced by the orientation of the circle action on X. This completes the proof of Theorem 1.

5. Applications and examples

5.1. **An application.** An immediate corollary to the main theorem is the calculation of the 3 dimensional Seiberg-Witten invariants for the total space of a circle bundle over a surface. The following corollary can also be derived from [MOY] using different techniques.

Corollary 8. *Let $\pi : Y \to \Sigma_g$ be a smooth 3-manifold which is the total space of a circle bundle over a surface of genus $g > 0$. Let $c_1(Y) = n\lambda \in H^2(\Sigma_g;\mathbb{Z})$ where λ is the generator. The only invariants which are not zero on Y come from* spin^c *structures which are pulled back $\pi : Y \to \Sigma_g$. Hence,*

$$SW_Y(\pi^*(s\lambda)) = \sum_{t\equiv s \mod n} SW_{\Sigma_g\times S^1}(\tilde{\pi}^*(t\lambda))$$

where $\tilde{\pi} : \Sigma_g \times S^1 \to \Sigma_g$.

Proof. Let $\pi : Y \to \Sigma_g$ be the total space of a circle bundle over Σ with Euler class $n\lambda$. Then the manifold $Y \times S^1$ can be thought of as a smooth 4-manifold with a free circle action which orbit space is $\Sigma_g \times S^1$. The Euler class of the action is $\tilde{\pi}^*(n\lambda))$. Applying the main theorem gives

$$SW^4_{Y\times S^1}((\pi, id)^*(\tilde{\pi}^*(s\lambda))) = \sum_{\tilde{\pi}^*(t\lambda)\equiv\tilde{\pi}^*(s\lambda) \mod \tilde{\pi}^*(n\lambda)} SW^3_{\Sigma\times S^1}(\xi')$$

the right hand side of the equation. Applying Proposition 7 shows that $SW^4 = SW^3$ in this case. ∎

Combining the Seiberg-Witten polynomial for the product of a surface with a circle,

$$\mathcal{SW}_{\Sigma_g\times S^1}(t) = (t^1 - t^{-1})^{2g-2},$$

with the previous results gives a formula for the Seiberg-Witten polynomial in terms of the Euler class and the genus of the surface.

Corollary 9. *Let $\pi : Y \to \Sigma_g$ be the total space of a circle bundle over a genus g surface. Assume $c_1(Y) = n\lambda$ where $\lambda \in H^2(\Sigma_g;\mathbb{Z})$ is the generator and n is an even number $n = 2l \neq 0$, then the Seiberg-Witten polynomial of Y is*

$$\mathcal{SW}_Y(t) \quad = \quad sign(n)\sum_{i=0}^{|l|-1}\sum_{k=-(2g-2)}^{k=2g-2}(-1)^{(g-1)+i+k|l|}\binom{2g-2}{(g-1)+i+k|l|}t^{2i}$$

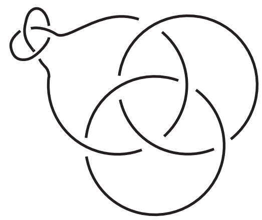

FIGURE 1. M_K before surgery.

where $t = exp(\pi^*(\lambda))$ *and defining the binomial cofficient* $\binom{p}{q} = 0$ *for* $q < 0$ *and* $q > p$*. For the formula where* n *is odd, replace* l *by* n *and* t^{2i} *by* t^i*.*

If one uses [MT] to calculate the Milnor torsion for a circle bundle Y over a surface, one finds that the invariant is identically 0. This is because all spin^c structures on Y with nontrivial invariants have torsion first Chern class. Turaev introduced another type of torsion in [Tu1, Tu2] and a combinatorially defined function on the set of spin^c structures $T : \mathcal{S}(Y) \to \mathbb{Z}$ derived from this torsion, and showed that this function was the Seiberg-Witten polynomial up to sign. Hence, principal S^1-bundles over surfaces provide simple examples which illustrate the difference between Milnor torsion and Turaev torsion.

5.2. **A construction and a calculation.** The following construction is similar to but simpler than the main construction in [FS2]. Let Y_K denote the manifold resulting from 0-surgery on a knot K in S^3. Let m be a meridian of the knot in Y_K. Let m_1, m_2, m_3 be loops that correspond to the S^1 factors of T^3. Construct a new manifold

$$M_K = T^3 \#_{m_1=m} Y_K = [T^3 \setminus (m_1 \times D^2)] \cup [Y_K \setminus (m \times D^2)]$$

by removing tubular neighborhoods of m and m_1 and fiber summing the two manifolds along the boundary such that $m = m_1$ and such that ∂D^2 is sent to ∂D^2.

This is a familiar construction. If one forms a link L from the Borromean link by taking the composite of the first component with the knot K (see Figure 1), then M_K is the result of surgery on L with each surgery coefficient equal to 0. If K is a fibered knot, then the resulting manifold $T^3 \#_{m_1=m} Y_K$ is a fibered 3-manifold.

Consider the formal variables $t_\beta = \exp(PD(\beta))$ for each $\beta \in H_1(M; \mathbb{Z})$ which satisfy the relation $t_{\alpha+\beta} = t_\alpha t_\beta$. The Seiberg-Witten polynomial $\mathcal{SW}$ of X is a Laurent polynomial with variables t_β and coefficients equal to the Seiberg-Witten invariant of the spin^c structure defined by t_β.

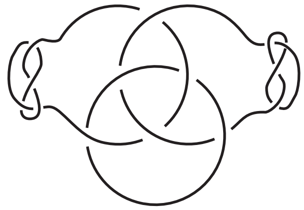

FIGURE 2. $M_{K_1K_2}$ before surgery

Theorem 10 (Meng and Taubes [MT]). *In the situation above*

$$\mathcal{SW}^3_{M_K} = \Delta_K(t^2_{m_1}) \tag{9}$$

where Δ_K is the symmetrized Alexander polynomial of K.

For example, the manifold M_K in Figure 1 where K is the trefoil knot has Seiberg-Witten polynomial

$$\mathcal{SW}^3_{M_K}(t_{m_1}) = -t^{-2}_{m_1} + 1 - t^2_{m_1}.$$

5.3. **Example 1.** We first produce an example of a nonsymplectic 4-manifold which admits a free circle action whose orbit space is a 3-manifold which is fibered over the circle. Our construction generalizes easily to produce a large class of such manifolds with this property. Let K_1 and K_2 be any fibered knots. Form the fiber sum of the complements of K_1 and K_2 with neighborhoods of the first and second meridians of T^3, i.e.,

$$M_{K_1K_2} = (S^3 \setminus K_1)\#_{m=m_1}T^3\#_{m_2=m}(S^3 \setminus K_2)$$

where m is the meridian of the corresponding knot. Since both K_1 and K_2 are fibered, the manifold $M_{K_1K_2}$ is a fibered 3-manifold. By Meng-Taubes theorem, the Seiberg-Witten polynomial of this manifold is

$$\mathcal{SW}^3_{M_{K_1K_2}}(t_{m_1}, t_{m_2}) = \Delta_{K_1}(t^2_{m_1})\Delta_{K_2}(t^2_{m_2}).$$

Let $X_{K_1K_2}(l)$ be the 4-manifold with free circle action that has $M_{K_1K_2}$ for its orbit space and $PD[l]$ for the Euler class of the circle action. Taking both K_1 and K_2 to be the figure eight knot (see Figure 2), we get a manifold with Seiberg-Witten polynomial:

$$\begin{aligned}\mathcal{SW}^3_{M_{K_1K_2}} = t^{-2}_{m_1}t^{-2}_{m_2} - 3t^{-2}_{m_2} + t^2_{m_1}t^{-2}_{m_2} - 3t^{-2}_{m_1} + 9 \\ - 3t^2_{m_1} + t^{-2}_{m_1}t^2_{m_2} - 3t^2_{m_2} + t^2_{m_1}t^2_{m_2}.\end{aligned}$$

The Seiberg-Witten polynomial of the manifold $X_{K_1K_2}(4m_1)$ can be calculated from Theorem 1,

$$\mathcal{SW}^4_{X_{K_1K_2}(4m_1)} = 2t^{-2}_{m_1+m_2} - 3t^{-2}_{m_2} + 9 - 6t^2_{m_1} + 2t^2_{m_1+m_2} - 3t^2_{m_2},$$

where $t_\beta = \exp(\pi^*(PD(\beta)))$ is the pullback of the spin^c structure on $M_{K_1K_2}$.

A theorem of Taubes [T] implies that the first Chern class c_1 of a symplectic 4-manifold must have Seiberg-Witten invariant ± 1. We thus see that the manifold $X_{K_1K_2}(4m_1)$ admits no symplectic structure with either orientation. This is not the only free S^1-manifold over $M_{K_1K_2}$ with this property. The manifolds $X_{K_1K_2}(-4m_1)$, $X_{K_1K_2}(4m_2)$, and $X_{K_1K_2}(-4m_2)$ also admit no symplectic structures.

5.4. **Example 2.** Next we produce an example of a 3-manifold which is not the orbit space of any symplectic 4-manifold with a free circle action. Let $K_1 = K_2$ be the nonfibered knot 5_2 (see [R]). The Seiberg-Witten polynomial of $M_{K_1K_2}$ is

$$\begin{aligned}\mathcal{SW}^3_{M_{K_1K_2}} = 4t^{-2}_{m_1}t^{-2}_{m_2} - 6t^{-2}_{m_2} + 4t^2_{m_1}t^{-2}_{m_2} - 6t^{-2}_{m_1} + 9 \\ - 6t^2_{m_1} + 4t^{-2}_{m_1}t^2_{m_2} - 6t^2_{m_2} + 4t^2_{m_1}t^2_{m_2}.\end{aligned}$$

One then needs to calculate as in Example 1. There are only finitely many free S^1 manifolds $X_{K_1K_2}(l)$ which need to be checked because for all $l = am_1 + bm_2$ with $|a|, |b| > 2$ the Seiberg-Witten polynomial $\mathcal{SW}^4$ is equal to the 3-dimensional polynomial (only the meaning of the variables will change). A calculation shows that the remaining free S^1-manifolds all have spin^c structures with Seiberg-Witten invariant greater than one in absolute value. Therefore these manifolds are not symplectic. Therefore $M_{K_1K_2}$ is not the orbit space of any symplectic 4-manifold with a free circle action.

5.5. **Remarks.** The above two examples show:

1. *There exist nonsymplectic free S^1-manifolds with fibered orbit space.*

2. *There exists a 3-manifold which is not the orbit space of any symplectic 4-manifold with a free S^1-action.*

We conclude with two questions.

Question 1. *If X is a free S^1-manifold which is symplectic, must its orbit space $M = X/S^1$ be fibered?*

Taubes has conjectured this in case $X = M \times S^1$. Theorem 1 could be used to search for manifolds with free S^1-actions that had nonfibered orbit spaces and which do not have Seiberg-Witten obstructions to having symplectic structures. One would still need to prove that those

manifolds where symplectic. While a counter example may be obtainable, a proof to the affirmative is already at least as difficult as a proof of Taubes' conjecture.

Question 2. *Let M be a 3-manifold with the property that every free S^1-manifold whose orbit space is M is symplectic. Is M fibered?*

The 3-torus is an example of manifold with this property [FGG].

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